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The elliptic curves in gauge theory, string theory, and cohomology

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Abstract: Elliptic curves play a natural and important role in elliptic cohomology. In earlier work with I. Kriz, these elliptic curves were interpreted physically in two ways: as corresponding to the intersection of M2 and M5 in the context of (the reduction of M-theory to) type IIA and as the elliptic fiber leading to F-theory for type IIB. In this paper we elaborate on the physical setting for various generalized cohomology theories, including elliptic cohomology, and we note that the above two seemingly unrelated descriptions can be unified using Sen's picture of the orientifold limit of F-theory compactification on $K3$, which unifies the Seiberg-Witten curve with the F-theory curve, and through which we naturally explain the constancy of the modulus that emerges from elliptic cohomology. This also clarifies the orbifolding performed in the previous work and justifies the appearance of the w_4 condition in the elliptic refinement of the mod 2 part of the partition function. We comment on the cohomology theory needed for the case when the modular parameter varies in the base of the elliptic fibration.

Keywords: Duality in Gauge Field Theories, String Duality, F-Theory.

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Contents

1. Introduction

The form-fields of string theory and M-theory are important in the study of the global structure of the these theories. In previous work with I. Kriz [1 − 3], elliptic cohomology theory was proposed to describe the fields of type II string theories and their partition functions, as refinements of the K-theoretic description of the fields [4, 5] and of the partition function [6]. In the twisted case [7], i.e. in the presence of the NSNS field H_3, the twisted K-theoretic description is discussed in [7] for type IIA, and an S-duality covariant description for type IIB using generalized cohomology refinements was proposed in [2]. For M-theory, in [8 − 10], a higher degree analog of K-theory was proposed. In both string theory and M-theory, several generalized cohomology theories were considered, including the theory of topological modular forms, TMF [11].

The type IIA K-theoretic partition function was constructed and matched by Diaconescu, Moore and Witten [6] with the Rarita-Schwinger and form-field part of the partition of M-theory [12] . The cancellation of the DMW anomaly given by the seventh Stiefel-Whitney class W_7 led to the interpretation [1] as a condition of orientation in elliptic cohomology. Consequently, an elliptically-refined partition function was proposed. In [13] part of this conjecture was verified. Elliptic cohomology comes equipped with an elliptic curve, defined over the coefficient ring of the theory, usually an integral polynomial in certain generators, identified as higher degree analogs of the Bott generator. Since elliptic cohomology appeared from rather physical considerations, namely the cancellation of an anomaly, it is expected that the structures that come with elliptic cohomology should

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correspond to some physical interpretation. For theories with one step higher [1] than K-theory, the generators have dimension two and six, a fact which was interpreted in [1] as corresponding, respectively, to the string and the NS5-brane, or to the M2-brane and the M5-brane in the M-theory limit. The elliptic curve of elliptic cohomology was consequently interpreted as corresponding to the elliptic curve arising from the intersection of the M2-brane and the M5-brane. This system reduces to strings on D-branes and so is natural to consider from the string theory point of view (see [14, 15]). The anomalies of such a system have been studied in [16].

On the other hand, an S-duality covariant approach to describe the fields of type IIB string theory was proposed in [2, 3]. In studying modularity in [3], the elliptic curve of elliptic cohomology was given another interpretation, namely as corresponding to the fiber of F-theory over type IIB string theory. TMF leads to anomaly-free modularity. However, this theory is not an elliptic cohomology theory, but is rather obtained from it by orbifolding, in the same way that one would get real K-theory KO from complex K-theory K. As the elliptic curve was taken to correspond to a 'physical' space, i.e. the elliptic fiber in F-theory, the orbifolding was also proposed to correpond to actual physical orbifolding in spacetime, as the one leading from type IIB string theory to type I theory.

However, several points remain to be clarified. First of all, the above two descriptions seem to be a priori unrelated, and even possibly incompatible. For instance, why would one expect that the elliptic fiber of F-theory would have anything at all to do with the intersection of M-branes in M-theory. Second, the description in terms of F-theory requires the modular parameter τ to be a field that varies in some base space of an elliptic fibration, whereas the modular parameter that appears in elliptic cohomology, and in TMF, is simply a constant complex number. Third, the construction of the mod 2 part of the elliptic partition function in [1] required the spacetime to be oriented with respect to EO, i.e. that the Stiefel-Whitney class w_4 vanishes, a condition which is more natural in type I and heterotic string theories. Fourth, the 'physical' elliptic curves are of course defined over $\mathbb{C}$, while the cohomology curve is defined over the coefficient ring of the theory. The aim of this note is to reconcile the above points among themselves, to make the previous picture in [3] compatible with the web of dualities, and to propose further generalizations.

For the first three items, we use Sen's description [17 − 19] of the compactification of F-theory on $K3$ in the orbifold limit in the moduli space where the surface appears as an orbifold of a four-torus. This automatically justifies the orbifolding done in [3] and makes the picture compatible with the web of dualities. The intersection of M2 and M5 in M-theory is a self-dual string, and such a configuration corresponds to the lift [20] of Seiberg-Witten theory [21, 22], i.e. $\mathcal{N} = 2$ $d = 4$ super-Yang-Mills theory, to M-theory. Sen also provided the embedding of Seiberg-Witten theory in F-theory using the same orbifold construction, where the mathematical resemblance of the two theories is striking and it is interesting that such a connection involves properties of the elliptic curves. The gauge theory is $\mathcal{N} = 2$ SYM with group $SU(2)$ and with four quark flavors. The moduli space of this theory was characterized by a gauge-invariant quantity u, and the complex

[1] in the so-called chromatic tower which will be explained in the next section.

coupling constant τ varies as we move in the u-plane [21, 22]. Sen [17] has shown that this is identical to the F-theory background with u labelling the coordinates of the base of the elliptic fibration and τ denoting the axion-dilaton modulus. The particular orientifold background above corresponds to the classical limit, which is singular since $Im(\tau)$ becomes singular in some regions. The identification is as follows

$$\begin{aligned}\text{gauge coupling} &\leftrightarrow \text{axion} - \text{dilaton}\\ \text{masses } m_i \text{ of the hypermultiplets} &\leftrightarrow \text{positions of } 7-\text{branes}\\ \text{the vev } z = \langle \text{tr } \phi^2 \rangle \text{ of the adjoint scalar} &\leftrightarrow \text{a point in the base (locally } \mathbb{R}^2/\mathbb{Z}_2).\end{aligned} \tag{1.1}$$

We start with a section on the general setting for generalized cohomology theories encountered in this paper and in the previous work. In particular we discuss the refinement of the Ramond-Ramond fields, (Hodge or electric/magnetic) duality symmetry and how it fits in this formalism, and the dynamics of M2- and M5-branes and their intersections. Part of this interpretation has appeared in [10]. We then review the description of the three elliptic curves and then relate them by Sen's orientifold pictures, making the required clarifications along the way and in the discussion section. Sen's picture was made more than a mathematical correspondence in [23], where it was shown that the picture can be captured by a probe D3-brane, representing the gauge theory, in the presence of D7-branes. We comment on this and on the consequence of deformation away from the limit to variable τ for the corresponding cohomology theory. We also interpret the formal parameter q as a coupling expansion parameter. While the arguments in this paper are mostly qualitative, we do provide in the discussion section some proposals for quantitative descriptions and further checks.

2. The physical setting for generalized cohomology

In this section we argue for a physical setting for various generalized cohomology theories by studying the structures involved. We look at the fields that appear in string theory and M-theory and identify which aspects of the fields are described by which cohomology theory. There are two issues: RR fields getting information about an elliptic curve, i.e. an elliptic refinement of the K-theoretic description and, second, Hodge duality between the string and the NS5-brane or between the M2-brane and the M5-brane. The second case involves complex-oriented generalized cohomology theories, i.e. ones descending from complex cobordism. It is important to distinguish elliptic vs. non-elliptic theories.

2.1 Interpreting the "generalized" in generalized cohomology

A *generalized* cohomology $\mathcal{H}$ is a theory that satisfies all the properties of a usual cohomology theory except for one, which is called the dimension axiom. This is the cohomology of a point, i.e. $\mathcal{H}^*(pt)$. This is also called the coefficient ring of the theory. For 'un-generalized' theories, this is just $\mathbb{Z}$. For the generalized theories this can be something much more complicated, e.g. a formal polynomials in certain parameters, with coefficients that can be integral or integral mod p.

Among the generalized cohomology theories in which we are interested and which have appeared in the study of type II string theories [1 − 3] — and to some extent also in M-theory [8 − 10] — are the theories in the so-called chromatic tower of spectra,[2] i.e. the ones that descend from complex cobordism theories. Those can be seen, looking from the other end, as a generalization of K-theory by alowing Bott generators of higher dimensions. An example of this type is (integral) Morava K-theories. In the previous work and in this paper, we also encounter a generalization of another type, namely *elliptic* cohomology theories which have an *elliptic* structure manifested by an elliptic curve parametrizing the coefficients in the cohomology of a point. This kind of cohomology theories will still contain the usual Bott element $v_1 = u$ of dimension two, but will have coefficients that are parametrized by the elliptic curve.

What is the implication of this for the physics? We will give some intuitive arguments on what the above means physically for us. We would like to interpret the fact that we have a nontrivial coefficient ring, which is the cohomology of a point, as indicating a nontrivial structure on (or over) that point. For example, this could be a generalization of a bundle structure. From the algebraic side, this is just having a sheaf structure over a given point. For us, this is close to what a scheme is, and so we see that this will perhaps justify the use of the concept of a *generalized elliptic curve* in such cohomology theories. The above implies that a point in this context has more structure than what one would usually associate to a point. In the particular case of elliptic cohomology, this structure will be a (family of generalized) elliptic curves. So, again intuitively, we have a somewhat hidden elliptic curve over each point. This is very similar to the Kaluza-Klein idea, in the sense that we see an internal structure over each point. This is yet another way of justifying the interpretation of the elliptic curve appearing in elliptic cohomology as corresponding to a physical elliptic curve in [1, 3].

2.2 Physical association of cohomology theories

We start by looking at a special class of generalized cohomology theories, namely elliptic spectra. An *elliptic spectrum* consists of [11]:

1. an even, periodic, homotopy commutative ring spectrum E with formal group P_E over the coefficient $E^0(pt)$;
2. a generalized elliptic curve $\mathcal{E}$ defined over $E^0(pt)$;
3. an isomorphism t of P_E with the formal completion $\hat{\mathcal{E}}$ of $\mathcal{E}$. [3]

The periodicity condition is explained by the following definition. A 2-*periodic* ring spectrum as a generalized cohomology theory E with an orientation of the complex line bundle on $\mathbb{C}P^\infty$ which, when restricted to $\mathbb{C}P^1$, has an inverse in E_*. The *evenness* condition means that all odd cohomology vanishes, i.e. $E_{2n+1} = 0$ for all n. This definition involves the notion of *generalized elliptic curves* over a ring R. This means a marked curve over

[2] For our purposes, the word "spectrum" simply means a generalized cohomology theory.

[3] Note that an elliptic curve is a one-dimensional abelian group.

the scheme Spec(R) [4] which is locally isomorphic to a Weierstrass curve, i.e.

$$\mathcal{E}: \ y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \tag{2.1}$$

with $a_1, a_2, a_3, a_4, a_6 \in R$. We will come back to the discussion of the elliptic curve and formal groups in the next section.

Ramond-Ramond fields: first we argue that, granted a justification for its use, the concept of an elliptic spectrum is not such a strange concept for physics. It is in fact already closely related to the discussion of Ramond-Ramond (= RR) fields in K-theory. Recall that in the K-theoretic description of the RR fields [4, 5]– say in type IIA– one has the total gauge-invariant field strength written as $F = \sum_{n=0}^{5} F_{2n}$. In order to make the RR field strengths homogeneous of degree zero, one can [24] use K-theory with coefficients in $K(pt) \otimes \mathbb{R} \cong \mathbb{R}[[v_1, v_1^{-1}]]$, where the inverse Bott element $v_1 \in K^2(pt)$ has degree 2; then the total RR field strength is written as the total uniform degree-zero expression

$$F = \sum_{m=0}^{5} (v_1)^{-m} F_{2m}. \tag{2.2}$$

Strictly speaking, we are dealing with cohomology rather than K-theory because we have already taken the image with respect to the Chern character included within the F's (see [7] or [10]).

Let us now rewrite this in a way that *resembles* an elliptic spectrum [11]. First note that there are three kinds of algebraic groups of dimension one: the additive group $\mathbb{G}_a$, the multiplicative group $\mathbb{G}_m$, and elliptic curves. The first one is just the complex plane $\mathbb{C}$ with the operation of addition on complex numbers and the second is $\mathbb{C}^*$ with the multiplication operation. An elliptic curve over $\mathbb{C}$ is of the form $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$. The map of formal groups derived from $\mathbb{C} \to \mathbb{C}/\Lambda$ gives an isomorphism t_Λ, from the additive formal group $\hat{\mathbb{G}}_a$ to the formal completion of the elliptic curve $\hat{\mathcal{E}}_\Lambda$. Let R_λ be the graded ring $\mathbb{C}[u_\Lambda, u_\Lambda^{-1}]$ with $|u_\Lambda| = 2$, and define an elliptic spectrum $H_\Lambda = (E_\Lambda, \mathcal{E}_\Lambda, t_\Lambda)$ by taking E_Λ to be the spectrum representing $H_*(-; R_\Lambda)$, $\mathcal{E}_\Lambda$ the elliptic curve $\mathbb{C}/\Lambda$, and t_Λ the isomorphism described above. So if we identify v_1 with u_Λ then in describing the RR fields as above we are dealing with an analog of the elliptic spectrum. Strictly speaking, it is not elliptic since this would require $n = 2$.

Among the examples given in [11] and used in [1] is the elliptic spectrum

$$E_* = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6][u, u^{-1}] \tag{2.3}$$

associated with the Weierstrass curve which we write in the form

$$y^2 + a_1uxy + a_3u^3y = x^3 + a_2u^2x^2 + a_4u^4x + a_6u^6, \tag{2.4}$$

where $\dim(x) = 4$ and $\dim(y) = 6$. Again, this connects with the RR fields since the coefficient ring (2.3) has u but yet also has information about the elliptic curve [5], since

[4] Intuitively, this is just a ‘thickened’ version of the original curve where the thickening can be understood as the structure over the points, (e.g. multiplicities). Supermanifolds, in one version, have such a description.

[5] which was interpreted in [3] as corresponding to the F-theory elliptic fiber, the further justification of which is one of the aims of this note.

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over each point we have such a fiber. Thus this is the most straightforward generalization of the above K-theoretic homogeneous degree zero description for the RR fields in type II string theory. We take this to correspond to refinements [3] leading to the RR formula

$$F(x) = \sigma(X)^{1/2} ch_E(x), \tag{2.5}$$

where $\sigma(X)$ is the Witten genus of the manifold X and ch_E is the Chern character of the theory E. This still has the same Bott generators as K-theory, but now instead of taking coefficients that are integers, we are taking them to be integral polynomials in the coefficients a_i of the Weierstrass curve.

Electric/magnetic duality: Among the interesting complex-oriented generalized cohomology theories obtained from complex cobordism (see [1] for an exposition) are Johnson-Wilson theory $BP\langle n\rangle$, Landweber elliptic cohomology $E(n)$, Morava K-theory $K(n)$ and integral Morava K-theory $\widetilde{K}(n)$. For $n = 2$, the coefficient rings of these and of complex K-theory are given as

$$\begin{aligned} BP_* &= \mathbb{Z}[v_1, v_2], \\ E(2)_* &= \mathbb{Z}[v_1, v_2, v_2^{-1}], \\ \widetilde{K}(2)_* &= \mathbb{Z}[v_2, v_2^{-1}], \\ K(2)_* &= \mathbb{Z}/p[v_2, v_2^{-1}], \\ K_* &= \mathbb{Z}[v_1, v_1^{-1}]. \end{aligned} \tag{2.6}$$

Such a 'hierarchy' of theories is sometimes referred to as (part of) the chromatic tower for spectra. The elements in the coefficients are related to the formal group laws. For $K(n)_*$ this has *height* n, which means that, in particular for $p = 2$ and 3 respectively,

$$\begin{aligned} [2]_F x &= x +_F x = v_n x^{2^n} \\ [3]_F x &= x +_F x +_F x = v_n x^{3^n}. \end{aligned} \tag{2.7}$$

We would like to comment on the structure of the above theories and propose their implication for physics. For $p = 2$, the dimensions of the generators are 2 and 6 for v_1 and v_2 respectively. In [1], $\widetilde{K}(2)$ and $E(2)$ were used, and there it was proposed that the generators correspond to worldvolumes of the corresponding intersection of membranes and fivebranes. From a IIA point of view, it was proposed that this was related to the fundamental string and the Neveu-Schwarz fivebrane, respectively. [6]

Here we would like to go further and propose the setting for the above theories in string theory and M-theory. Since v_1 corresponds to the (boundary of the) membrane and v_2 corresponds to the fivebrane, and since those in turn couple to the field C_3 and to the potential C_6 of the dual of its field strength, respectively, then the v_i $(i = 1, 2)$ should tell us something about Hodge duality. So in some sense, generalized cohomology theories that

[6]Strictly speaking, the following discussion for $p = 2$ is better adapted for string theory, and in M-theory one might need to go [10] to $p = 3$, or better yet, to TMF where all primes are treated democratically. In any case, the discussion can be made to be generic.

contain only v_1 are 'electric', theories that contain only v_2 are 'magnetic', and theories that contain both v_1 and v_2 are 'electro-magnetic'. Therefore, by inspecting (2.6), we propose the following identifications between generalized cohomology theories and physical theories,

$$\begin{aligned} \mathrm{K-theory} &\longleftrightarrow \mathrm{non-dual\ theory} \\ \mathrm{(integral)\ Morava\ K-theory} &\longleftrightarrow \mathrm{dual\ theory} \\ \mathrm{Brown-Peterson\ theory} &\longleftrightarrow \mathrm{duality-symmetric\ theory} \\ \mathrm{Landweber\ theory} &\longleftrightarrow \mathrm{special\ duality-symmetric\ theory.} \end{aligned}$$

The last one is termed "special" because of the fact that the second generator, v_2, is inverted while the first one, v_1, is not, and so this suggests a more privileged role for the dual field. In light of the identifications we have done, namely the brane worldvolumes with the generators, and the generalized cohomology theories with types of physical theories with respect to duality, we further propose that K-theory is a theory that has to do with membranes (or strings, or electric fields), Morava K-theory with fivebranes (or magnetic fields), Brown-Peterson theory with membranes and fivebranes, and Landweber's theory with membranes and fivebranes and where the latter play a special role since the generator corresponding to them, v_2, is inverted. One way to explain this inversion is to say that the M5-brane and the NS5-brane are more fundamental that the M2-brane and the string. This might be justified either by a form of the Hanany-Witten effect [25] adapted for this case, where, for example, an M2-brane is created out of an M5-brane [7] or by saying that fivebranes are instinsically more fundamental than either the membrane or the string.

There are two aspects to explain. First, that D-branes and strings can be obtained from the M-branes, and second, that the M5-brane in some sense already encodes the M2-brane. Since M-theory does not have a perturbative coupling constant analogous to the dilaton in string theory, the relationship between M2 and M5 (and at the level of fields, between C_3 and C_6) should be different from the usual strong-weak coupling duality between strings and NS5-branes in ten dimensions. Based on this, it was argued in [27] that the M2-brane may be a limiting case of the M5-brane, which would imply that M-theory is self-dual. One way to see this is to note that [27] besides the dual six-form C_6 of eleven-dimensional supergravity, the five-brane also couples directly to the three-form C_3 itself, via an interaction of the type

$$\int C_3 \wedge T_3, \tag{2.8}$$

where T_3 is a self-dual three-form field strength that lives on the M5-brane world-volume. Maps between the generalized cohomology theories are characterized by the corresponding maps, if they exist, between the coefficient rings. There are maps from BP to all other descendents, and also from $BP\langle n\rangle$ to $E(n)$, but there are no maps between the $K(n)$'s for different n. This has implications on duality, because for us this implies that one cannot relate $K(1)$ information, i.e. usual K-theoretic information from v_1, to the 'dual' $K(2)$ information, i.e. the Morava K-theoretic information from v_2, of course unless we are already in the context of a bigger theory that contains both theories.

[7]Some discussion on this can be found in [26, 10].

M2-M5 states and interactions: by allowing this field T_3 to have non-trivial fluxes through the three-cycles on the world-brane, the five-brane can thus in principle carry all membrane quantum numbers. These configurations are therefore naturally interpreted as bound states between the two types of branes. Based on this the authors of [27] argue that the M5-brane is a natural candidate to give a more unified treatment of all BPS states in string theory. Indeed, they give a unified description of all BPS states of M-theory compactified on T^5 in terms of the M5-brane. An important ingredient in this formalism is the idea that the relevant degrees of freedom on the five-brane are formed by the ground states of a string theory living on the world-volume itself. In [27] it was further argued that by comparing to the analysis of D-brane states [28], it should be possible to make concrete identification between specific five-brane excitations and configurations of D-branes and fundamental strings. Via this correspondence our arguments on the BPS spectrum should also give useful information about the bound states of strings and D-branes [29].

What we are advocating is that the generalized cohomology theories that we are dealing with have the correct mathematical structure to incorporate the interactions between M2-branes and M5-branes, which, in light of the arguments in [27], means that this also incorporates bound states and interaction of type IIA string theory (and type IIB by T-duality). In particular, this suggests identifying the parameter q that appeared in the refinement of the type IIA and M-theory partition functions in [1] as a coupling constant which encodes the interactions of M2-M5 systems. Then, this would also imply that this coupling constant also encodes the interactions of strings and branes in ten dimensions.

3. The elliptic curve in elliptic cohomology

We start with discussing the (in-)dependence of the generalized cohomology theories on the primes p. In string theory, usually the prime 2 is important, i.e 2-torsion appears in the fields, and this can be seen for example in the K-theory calculation of the IIA partition function in [6]. A notation like $\mathbb{Z}[\frac{1}{2}]$ means that 2 is inverted, which implies that we stay away from 2-torsion. In contrast, when we say we localize at 2, this means that we are looking at 2-torsion, i.e. we cannot freely divide by 2. Then, from the point of view of the chromatic tower, this means that we are looking at $K(1)$-local information since this contains v_1. Of course the same analysis holds for the other primes. There is a simplification that occurs in the Weierstrass equation (and the definition of E), but that is at the expense of losing torsion information since it requires inverting of 2 and 3.

For $n = 2$ in (2.7), one can also consider cohomology theories whose formal group laws are elliptic, i.e. are obtained by Taylor expansion of the group law on an elliptic curve over some commutative ring. Such theories are the *complex-oriented elliptic cohomology theories*. For the sake of exposition, let us briefly see how this works (this is standard mathematical literature, see e.g. [30] or [31]). For (2.1), introduce new variables $t = \frac{-x}{y}$ and $s = \frac{-1}{y}$. By iteratively solving for s in terms of t, one can write x and y in terms of the variable t only, i.e.

$$x = t^{-2} - a_1 t^{-1} - a_2 - a_3 t - (a_4 + a_1 a_5)t^2 + \cdots , \tag{3.1}$$

and y is just $-x/t$. We see that x and y are power series expansions in the variable t and with coefficients given by the a's, and so they are in $\mathbb{Z}[a_1, a_2, a_3, a_4, a_6][[t]]$. [8]

The group law can be obtained by working near $t = 0$ (t identified as a local parameter near the origin $(t, s) = (0, 0)$ in E). If $(t_1, s_1) + (t_2, s_2) = (t_3, s_3)$ on the elliptic curve E in the (t, s)-plane, then $t_3 \equiv \Phi(t_1, t_2)$ has the form

$$t_3 = t_1 + t_2 - a_1 t_1 t_2 - a_2(t_1^2 t_2 + t_1 t_2^2) - 2a_3(t_1^3 t_2 + t_1 t_2^3) + \cdots , \tag{3.2}$$

and so $\Phi(t_1, t_2)$ similarly is in $\mathbb{Z}[a_1, a_2, a_3, a_4, a_6][[t_1, t_2]]$. If the coefficients a_j of E lie in a ring R, then $t_3 = \Phi(t_1, t_2)$ is in $R[[t_1, t_2]]$. The formal series $\Phi(t_1, t_2)$ arising from the group law on E is a *formal group law* FGL. This is given by

$$\hat{C}(x, y) = x + y - a_1 xy - a_2(x^2 y + xy^2) - (2a_3 x^3 y - (a_1 a_2 - 3a_3)x^2 y^2 + 2a_3 xy^3) + \cdots . \tag{3.3}$$

The FGL can also be defined as the way line bundles behave under tensor product,

$$\hat{G}_E(x, y) = e(L_1 \otimes L_2) \in E^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong \pi_* E[[x, y]], \tag{3.4}$$

where e is the Euler class.

There are various models for complex-oriented cohomology E which are characterized by their coefficient rings $E^*(pt)$. Choosing a complex-oriented elliptic cohomology theory corresponds to choosing coordinates on the elliptic curve. The Weierstrass curve is the universal (generalized) elliptic curve, and is given by the equation

$$y^2 x + a_1 xyz + a_3 y^3 = x^3 + a_2 x^2 z + a_4 xz^2 + a_6 z^3$$

over the ring $\mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$, where the parameters a_i are (generalized) modular forms. However, this curve has automorphisms and thus the corresponding theory with $E_* = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6][u, u^{-1}]$ is not universal.

Reduction of coefficients: in the physical cases, i.e. in Seiberg-Witten theory and in F-theory, the elliptic curves are physical parts of spacetime, i.e. are looked at as Riemann surfaces. Thus they are defined over $\mathbb{C}$. On the other hand, the elliptic curves that show up in elliptic cohomology are defined over various different fields or rings. This includes finite fields $\mathbb{F}_p$, integral or p-integral polynomial rings. Thus, at a first glance, the two pictures seem to be incompatible. However, there is a standard procedure to get curves over the latter coefficients starting with curves over $\mathbb{C}$. [9]

From an elliptic curve over $\mathbb{C}$ one can get an elliptic curve over $\mathbb{Z}$ if one can define the curve over $\mathbb{Q}$. The latter of course is not a serious condition and is in some sense the 'default' situation. One can get the curve over $\mathbb{Z}[x_1, x_2, \cdots]$ or $\mathbb{Z}[t]$ if one can define the curve over some finite extension of $\mathbb{Q}$, again not a very serious condition. In all the relevant cases, the resulting curve would have the same Weierstrass form, except that the

[8] A notation like $f((t))$ means power series in t, while $f[[t]]$ means restriction to non-negative powers.

[9] The question of whether the resulting 'discrete' varieties correspond *in reality* to some discrete versions of spacetime seems to be a much deeper question that goes far beyond the scope of this paper.

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coefficients a_i, instead of taking values in $\mathbb{C}$, now take values in the new field or ring. For example, for $\mathbb{Z}[t]$, one has $a_i(t) \in \mathbb{Z}[t]$, and

$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t), \tag{3.5}$$

giving an elliptic curve E over $\mathbb{Z}[t]$.

4. The elliptic curve in Seiberg-Witten theory

For Seiberg-Witten theory [21, 22], one can have any genus g for the curve and not necessarily just the elliptic case $g = 1$. The latter is the case for the originally considered gauge group $SU(2)$ only. In general, one can have any genus for the curve. In particular for groups of type A_n, the genus is equal to n [32, 33]. All ADE groups have also been considered, leading to different genera (greater than one) [34 − 36]. So one might ask the question of whether elliptic curves play any particularly important role in in the $N = 2$, $d = 4$ gauge theory.

The Seiberg-Witten gauge theory results can be obtained from type IIA string theory. In [36] The $N = 2$ $SU(2)$ SYM can be regarded as the worldvolume theory of parallel D4-branes. The D4's have a finite extent in one direction, along which they end on NS5-branes. it was shown how to obtain this genus g Riemann surface Σ_g from string theory on a Calabi-Yau threefold of the form $\Sigma_g \times \mathbb{R}^4$, which represents a symmetric fivebrane. The result was that the Riemann surface is given a concrete physical meaning in [36], where the BPS states correspond to self-dual strings (cf. [37]) that wind geodesically around the homology cycles, viewed as boundaries of D2-branes ending on the curve part of the fivebrane. This D2-brane can be viewed as the the disk formed from filling the one-cycle on the Riemann surface, giving the string on the Riemann surface as a boundary.

Witten [20] provided a derivation of Seiberg-Witten from M-theory and provided a geometric interpretation for the curve. The corresponding configuration is given in terms of single M5-brane with worldvolume $\mathbb{R}^4 \times \mathcal{E}$ with $\mathcal{E}$ an elliptic curve embedded in the Euclidean space X spanned by x^4, x^5, x^6, x^{10}, where x^{10} is the eleventh direction, i.e. the would-be 'M-theory circle', and $\mathbb{R}^4$ spanned by x^0, x^1, x^2, x^3. This space is endowed with a complex structure such that $s = x^6 + ix^{10}$ and $v = x^4 + ix^5$ are holomorphic with the requirement that $\mathcal{E}$ is a holomorphic curve in X of degree 2 in v.

The BPS states in the fivebrane worldvolume field theory correspond to M2-branes ending on the M5-brane [14, 15]. The boundary of M2 has to lie on $\mathcal{E}$ and it couples to the self-dual two-form on the world-volume of M5. The mass of the BPS states is given by that of the M2, which in turn is given by the brane tension times the area

$$m = T_2 \int_{M2} |\omega|, \tag{4.1}$$

where the area is given by the pullback of the holomorphic 2-form $\omega = ds \wedge dv$ [38]. Since $\partial M2 = \mathcal{E}$,

$$m^2 = T_2 \left| \int_{\partial M2} v(t) \frac{dt}{t} \right|^2 \tag{4.2}$$

where v is written as a function of $t = e^{-s}$. [10] Thus the mass of M2 reduces to an integral of a meromorphic one-form on $\mathcal{E}$, and the Seiberg-Witten differential is given by [38]

$$\lambda_{SW} = v(t)\frac{dt}{t}. \tag{4.3}$$

From the point of view of the self-dual string, this is just the tension [36].

One gets different elliptic curves depending on whether or not there is matter, and on the kind of matter allowed. After including hypermultiplets, the Coulomb branch of vacua is still parametrized by a copy of the u-plane (where u is related to $\langle \mathrm{Tr}\ \phi^2 \rangle$ in the underlying theory), but now the u-plane parametrizes a different family of elliptic curves. The appropriate families, which depend on the hypermultiplet bare masses, have the form [21, 22]: [11]

$$y^2 = x^3 + a_2 x^2 + a_4 x + a_6, \tag{4.4}$$

where a_2, a_4, a_6 are polynomials in u and in the masses m_i. They are also polynomials in the scale Λ of the theory for $N_f < 4$, or of certain modular functions $e_i(\tau_0)$ for $N_f = 4$ or for $\mathcal{N} = 4$, where τ_0 is the coupling as measured at $u = \infty$ in the $N_f = 4$ or $\mathcal{N} = 4$ theory.

5. The elliptic curve in F-theory

F-theory emerged [39] in order to explain geometrically the axion-dilaton combination in type IIB string theory in the Kaluza-Klein spirit as moduli of an internal torus. These moduli are not fixed but vary. One takes this to be an elliptic fibration over a base manifold. The complex structure modulus $\tau = \chi + ie^{-\phi}$ of the fiber then varies on the upper half plane. Consequently, as we move along closed cycles of the base, τ will undergo nontrivial $SL(2;\mathbb{Z})$ transformations.

Let us see how F-theory is related to M-theory. Recall that we have that the interpretation of type IIB in terms of F-theory requires the elliptic curve to have a modulus that varies on a base, which is taken to be $\mathbb{C}\mathrm{P}^1 = S^2$ for two-folds. To relate these F-theory compactifications to other string theories, one can further compactify on S^1 with radius r_{IIB}. This is equivalent to M-theory compactified on a two-torus T^2 [42]. Taking the tenth and eleventh radii to be r_9 and r_{10} respectively, the relation is

$$r_{IIB} = (r_9 r_{10})^{-3/4}. \tag{5.1}$$

So small type IIB S^1 corresponds to large area T^2 in M-theory, and so the supergravity approximation can be trusted.

One can also consider what happens to the modular parameter τ. Since the coupling in type IIB is given by

$$g_{IIB} = (r_{10}/r_9)^{1/2}, \tag{5.2}$$

[10] written this way since the eleventh direction is a circle.

[11] By reduction (see e.g. [30]) the curve (4.4) is equivalent to the curve: $y^2 = x^3 - \frac{c_4}{48}x - \frac{c_6}{864}$ with $c_4 = 16(a_2^2 - 3a_4)$ and $c_6 = -64a_2^3 + 288\ a_2\ a_4 - 864\ a_6$.

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τ will capture the conformal class of the metric on T^2, which is equivalent to specifying a holomorphic structure [40]. So there is a duality in seven dimensions between F-theory on $K3 \times S^1$ and M-theory compactified on a four-manifold which is fibered by T^2's with holomorphic structure dictated by $\tau(z)$. The four-manifolds of this kind are not unique, but there *is* a unique one for which the holomorphic fibration has a holomorphic section [40]. The above suggests the duality[39, 17] between M-theory on elliptically fibered manifold with a holomorphic section and F-theory from $\tau(z)$, further compactified on S^1. A Wilson line has to be turned on along the S^1 if the four-manifold has no holomorphic section [41, 40].

6. Relating the curves

Motivated by Heterotic/F-theory duality, Sen [17] studied F-theory over a $K3$-surface in the special point in the $K3$ moduli space where the surface is the $\mathbb{Z}_2$-orbifold of the four-torus T^4, corresponding to the case where the axion-dilaton modulus is constant. Such a background was identified with an orientifold of type IIB string theory. This configuration of F-theory on $K3$ is T-dual to type I string theory on T^2, which in turn is equivalent to heterotic string theory on T^2. This provided an embedding in F-theory of Seiberg-Witten theory, i.e. $N = 2$ SYM with group $SU(2)$ and with four quark flavors, whose moduli space can be characterized by a gauge-invariant quantity u, and the complex coupling constant τ that varies as one moves in the u-plane [21, 22]. Sen [17] has shown that this is identical to the F-theory background with u labelling the coordinates of the base of the elliptic fibration and τ denoting the axion-dilaton modulus. And so, for constant τ in both cases, one has a constant coupling in SW theory and a constant axion-dilaton pair in string theory. The gauge theory solution provides the dependence of τ on z and the parameters. Furthermore [17], the masses of BPS states can be expressed in F-theory in terms of period integrals of the holomorphic 2-form on the $K3$-surface. The BPS states become massless precisely when one or more of these integrals vanishes and so the surface becomes *singular*.

An elliptically fibered $K3$ surface can be constructed out of the Weierstrass form of an elliptic curve by letting the parameters a_i become polynomials in the $S^2 = \mathbb{C}\mathrm{P}^1$ coordinate z, and thus has the form

$$y^2 = x^3 + f(z)x + g(z) \tag{6.1}$$

where x, y and z are coordinates on the base $\mathbb{C}\mathrm{P}^1$, and $f(z)$ and $g(z)$ are polynomials in z of degree 8 and 12 respectively. This describes a torus for each point on $\mathbb{C}\mathrm{P}^1$ labelled by the coordinate z. The modular parameter $\tau(z)$ of the torus is determined in terms of the ratio f^3/g^2 through the relation to the j-invariant

$$j(\tau(z)) = \frac{4.(24f)^3}{27g^2 + 4f^3}. \tag{6.2}$$

The compactification of F-theory on this particular $K3$-surface corresponds to compactification of type IIB string theory on $\mathbb{C}\mathrm{P}^1$ labelled by z, with the modular parameter given by the axion-dilaton pair,

$$\tau(z) = \chi(z) + ie^{-i\phi(z)/2}. \tag{6.3}$$

Sen also gave the description in terms of D-branes. From this point of view, such a background corresponds to a configuration of 24 D7-branes transverse to the $\mathbb{CP}^1$ and situated at the zeroes of the discriminant $\Delta \equiv 4f^3 + 27g^2$. In terms of the positions z_i of the D7-branes, this is $\Delta = \prod_{i=1}^{24}(z - z_i)$. These singular fibers correspond to $j(\tau(z_i)) \to \infty$. Then considering a special point in the moduli space where $\tau(z)$ is independent of z requires from (6.2) that the ratio f^3/g^2 is a constant, which means that f and g can be written in terms of one ploynomial ϕ of degree 4 in z as $g = \phi^3$ and $f = \alpha\phi^2$, where α is a constant. From the point of view of D-branes, this corresponds to grouping the D7-branes into 4 sets of 6 coincident D7-branes, situated at the points z_i $i = 1, \cdots, 4$ where ϕ vanishes. The effect of this is that [17] the resulting base is the orbifold T^2/I_2 of the two-torus T^2 by the group that inverts the signs of both coordinates. This comes from an $SL(2;\mathbb{Z})$ monodromy diag$(-1,-1)$ around each of the points z_i.

The above transformation can be identified [17] with the discrete transformation $(-1)^{F_L}$ Ω of type IIB, where $(-1)^{F_L}$ changes the sign of all the Ramond sector states on the left moving sector and Ω is the orientation-reversal transformation that exchanges the left- and the right moving modes on the worldsheet. This means that one is considering type IIB string theory compactified on T^2/I_2 such that when one moves once around each point on T^2/I_2 the theory comes back to itself transformed by the symmetry $(-1)^{F_L}\Omega$. In other words, the theory can be identified with type IIB on T^2, modded out by the $\mathbb{Z}_2$ transformation $(-1)^{F_L}\Omega I_2$. This is a type I$'$ orientifold [43, 44] which is related to type I theory by a T-duality transformation. By making an $R \to 1/R$ transformation on both circles of T^2 one maps the $\mathbb{Z}_2$-transformation $(-1)^{F_L}\Omega I_2$ to the transformation Ω [45, 17]. The corresponding D7-brane configurations is identical to the one obtained from F-theory. This in turn establishes [17] the conjectural [46−49] equivalence to heterotic string theory on T^2.

Note that there are also other branches of constant τ. In addition to Sen's case considered above, there are two other branches [50]. First, $f = 0$, $g \neq 0$, corresponding to $\alpha \to \infty$, and second, $f \neq 0$, $g = 0$, corresponding to $\alpha \to 0$. The constant value of τ is i for the first case and $\exp(i\pi/3)$ for the second case. The j-invariant is 13824 for the first case and 0 in the second case. These corresponding discriminants are $\Delta = 4f^3$ and $\Delta = 27g^2$, respectively. The periods of $K3$ will then be given by

$$\Omega_i = C(\tau_i) \int \frac{dz}{\Delta^{1/12}}, \tag{6.4}$$

where $C(\tau_i)$ are constants corresponding to the constant values τ_i equal to τ_0, i and $e^{\frac{i\pi}{3}}$. Going back to the orbifold picture, these correspond to the $K3$ being $T^4/\mathbb{Z}_n$ where $n = 4$ in the first branch and $n = 3, 6$ in the second branch [50]. It is interesting that singularities of type E_6, E_7 and E_8 were obtained for $n = 3, 4$ and 6 respectively, which suggests that a description through weakly coupled D-branes is not possible even in a limit [50].

We will look at the picture in terms of the moduli space of $K3$. The $\mathbb{Z}_n$ orbifold construction of $K3$ can be described as follows [51]. Consider a four-torus T, where $T = T^2 \times \widetilde{T}^2$ with two $\mathbb{Z}_n$ symmetric two-tori $T^2 = \mathbb{C}/L$, $\widetilde{T}^2 = \mathbb{C}/\widetilde{L}$ which need not be orthogonal. Let $\zeta \in \mathbb{Z}_n$ act algebraically on $(z_1, z_2) \in T^2 \times \widetilde{T}^2$ by $(z_1, z_2) \mapsto (\zeta z_1, \zeta^{-1} z_2)$. Next mod out this symmetry and blow up the resulting singularities; that is, replace each singular point by a chain of exceptional divisors, which in the case of $\mathbb{Z}_n$-fixed points have as intersection matrix the Cartan matrix of A_{n-1}. In particular, the exceptional divisors themselves are rational curves, i.e. holomorphically embedded spheres with self intersection number -2. For $n \in \{2, 3, 4, 6\}$ this procedure changes the Hodge diamond by

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array} \longmapsto \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

and indeed produces a $K3$ surface X. One also obtain a rational map from T to X of degree n by this procedure. What is interesting is that the values of n in this construction are exactly the ones corresponding to the orbifold limit [17, 50], which makes the above picture consistent. Note that corresponding to the above geometric orbifolds are orbifold conformal field theories, whose moduli spaces are constructed in [52].

7. Discussion and proposals

Orbifolding and the w_4 condition. We would like to explain the appearance of the w_4 condition in the analysis of [1] in a way that makes it seem less foreign. There, one condition for constructing the mod 2 part of the elliptically refined partition function of type IIA string theory was the vanishing of the fourth Stiefel-Whitney class of spacetime $w_4 = 0$. It is known that this condition is related to type I and heterotic theories rather than type II string theories. So, if we interpret the elliptic curve of elliptic cohomolgy as that of F-theory, and we further go to the orbifold limit of Sen, then we naturally connect to the latter two string theories. This more explicitly explains the proposal in [1] that we see a unification of the various string theories when viewed through the eye of elliptic cohomology.

Explicitly relating the curves. There are three elliptic curves considered in this note. Our arguments in this paper were mostly on the relation between the two 'physical' elliptic curves (defined over $\mathbb{C}$), namely the Seiberg-Witten and the F-theory curves. This is just Sen's picture of the orbifold limit of $K3$. Combined with the previous work [1, 3] this suggests that all three curves are related in a precise way. However, we did not make any quantitative claims or checks in this note. One way of doing so is to relate the corresponding coefficients of the three elliptic curves. In Seiberg-Witten theory, the a_i are given by polynomials in the masses of the BPS states, and hence in charges for those states. This is related to BPS states in the M-brane configurations. It would be very interesting to make such a match quantitatively, and to (re)produce formulae for BS states. Sen has shown that there is a consistent map between the orientifold and the F-theory, $m_i = c_i$,$1 \leq i \leq 4$, where m_i label the F-theory and the parameters c_i label the orientifold.

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The q-expansions. The interpretations in this note also unify the emerging loop expansions in the M2-M5 system (and hence in the gauge theory) and in the F-theory picture. What we are advocating is that the generalized cohomology theories that we are dealing with have the correct mathematical structure to incorporate the interactions between M2-branes and M5-branes, which, in light of the arguments in [27], means that this also incorporates bound states and interaction of type IIA string theory (and type IIB by T-duality). In particular, this suggests identifying the parameter q that appeared in the refinement of the type IIA and M-theory partition functions in [1] as a coupling constant which encodes the interactions of M2-M5 systems. Then, this would also imply that this coupling constant also encodes the interactions of strings and branes in ten dimensions. This suggests the picture

$$K(X) \xrightarrow{\text{strong coupling}} K[[q]](X) \tag{7.1}$$

in such a way that taking the weak coupling limit just sends $q \to 0$, and we go back from the RHS to the LHS. The interpretation of q as a coupling constant is perhaps close to that in topological string theory. In particular, it would be interesting to provide a connection to Gromov-Witten invariants.

Reduction of coefficients. One might use the standard procedure in (mathematical) gauge theory of working over finite fields and then translating back to the base field considered. [12] Starting from the elliptic curve $\mathcal{E}$, we get the reduced curve $\overline{\mathcal{E}}$ where the coefficients a_i are reduced to $\overline{a}_i$ that take values in the finite field, via what is called the reduction homomorphism r_p from the original field to the finite field. The discriminant of $\overline{\mathcal{E}}$ is $\overline{\Delta}$, the reduction mod p of the discriminant Δ of $\mathcal{E}$. Clearly $\overline{E}$ is nonsingular if and only if $\overline{\Delta} \neq 0$. The elliptic curves are divided into curves with bad reduction and curves with good reduction, depending on whether the discriminant of the reduced curve is zero or not. $\mathcal{E}$ has good (bad) reduction at p provided that $\overline{\mathcal{E}}$ is nonsingular (singular) at p. In the former case, the reduction function r_p is a group morphism. In the latter this means that $\overline{\mathcal{E}}$ is not an elliptic curve. Further, $\mathcal{E}$ has additive reduction provided that the singularity in $\overline{\mathcal{E}}$ is a cusp, and $\mathcal{E}$ has multiplicative reduction provided that $\overline{\mathcal{E}}$ has a node. What is the point of working over finite fields? This is part of some deep connections to number theory. However, for us we are just using the correspondence between theories defined on continuous base and theories defined over finite fields. This is quite useful in other situations, for example in gauge theory, where striking analogies are drawn between the two pictures, as well as the utility of working over the finite fields to perform explicit calculations of physically interesting quantities.

Deforming away from orbifold limit. In this paper we have restricted to constant value of the modular parameter. This fits nicely with the elliptic cohomology picture. However, the physics allows for more, namely for τ which is a field, which can be viewed as a deformation away from the orbifold limit [17]. In F-theory such deformations correspond

[12] One might argue that, in our context, this is more than just a tool since the finite fields appear in some versions of elliptic cohomology.

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to splitting the 6 coincident zeros of Δ away from each other. From the orientifold point of view, this corresponds to moving the 4 coincident D7's away from the orientifold plane. Curiously, in the presence of D7-branes, Δ is not inverted because the positions of the D7-branes correspond to zeros of Δ; so one has to take that into account in the elliptic cohomology picture, i.e. if Δ is inverted in E then the theory cannot contain D7-branes, and conversely, if we want to describe D7-branes together with the singularities then we cannot invert Δ. The asymptotic value of the variable τ is taken to be the special value τ_0. The deformation away from the orbifold limit is described by a surface of the form

$$y^2 = x^3 + \widetilde{f}(z)x + \widetilde{g}(z), \tag{7.2}$$

where now $\widetilde{f}$ and $\widetilde{g}$ are polynomials in z of degree 2 and 3 respectively. This gives five complex paramters after removing one by an overall shift of z and another by a rescaling of x and y. The corresponding five parameters in Seiberg-Witten theory are given by the 4 quark masses m_i $(i = 1, \cdots, 4)$ and the complex coupling constant $\tau_0 \equiv (\frac{\theta}{2\pi} + \frac{4\pi i}{g^2})$. The value of τ is [17]

$$\tau(z) = \tau_0 + \frac{1}{2\pi i}\left(\sum_{i=1}^{4} \ln(z - z_i) - 4\ln(z)\right), \tag{7.3}$$

where z_i is m_i^2 or c_i^2 for Seiberg-Witten and the orientifold theory respectively

In this respect then, the full physical theory requires more than elliptic cohomology, i.e. something that has more aspects of $K3$. We might say that elliptic cohomology is at a special point in the moduli space of $K3$, and the required theory contains a generalization of the elliptic curve. Such a generalization would be a cohomology theory built out of $K3$, e.g. a $K3$-cohomology. From considerations in arithmetic algebraic geometry, one can build formal group laws corresponding to $K3$, and so in principle such a theory could exist. However, we do not know of a construction. In fact Sen's arguements can be generalized from $K3$ to higher dimensional Calabi-Yau spaces [18, 19], and thus hints at room even for Calabi-Yau cohomology (beyond $K3$). Further, at weak coupling and even away from the special point in the moduli space, the orientifold and the F-theory descriptions coincide. However, at strong coupling the orientifold description breaks down near the orientifold point. This implies that [17] F-theory provides the correct description of the background field configuration of this theory, and the orientifold background must be modified by *quantum corrections* so as to coincide with the F-theory background, and the description is nonperturbative since one has effects of the form $\exp(i\pi\tau_0/2)$. Consequently, the full F-theory limit was proposed to describe the quantum-corrected version of the orientifold background. For us, this suggests the conjecture that in order to capture the full quantum theory, elliptic cohomology needs to be corrected by some higher cohomology theory, which is either a version of elliptic cohomology adapted to elliptic fibrations i.e. which encodes variation in the moduli space, or a form of Calabi-Yau cohomology.

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References

[1] I. Kriz and H. Sati, *M-theory, type-IIA superstrings and elliptic cohomology*, *Adv. Theor. Math. Phys.* **8** (2004) 345 [hep-th/0404013].

[2] I. Kriz and H. Sati, *Type IIB string theory, S-duality and generalized cohomology*, *Nucl. Phys.* **B 715** (2005) 639 [hep-th/0410293].

[3] I. Kriz and H. Sati, *Type II string theory and modularity*, *JHEP* **08** (2005) 038 [hep-th/0501060].

[4] G.W. Moore and E. Witten, *Self-duality, Ramond-Ramond fields and K-theory*, *JHEP* **05** (2000) 032 [hep-th/9912279].

[5] D.S. Freed and M.J. Hopkins, *On Ramond-Ramond fields and K-theory*, *JHEP* **05** (2000) 044 [hep-th/0002027].

[6] D.-E. Diaconescu, G.W. Moore and E. Witten, *E_8 gauge theory and a derivation of K-theory from M-theory*, *Adv. Theor. Math. Phys.* **6** (2003) 1031 [hep-th/0005090].

[7] V. Mathai and H. Sati, *Some relations between twisted K-theory and E_8 gauge theory*, *JHEP* **03** (2004) 016 [hep-th/0312033].

[8] H. Sati, *M-theory and characteristic classes*, *JHEP* **08** (2005) 020 [hep-th/0501245].

[9] H. Sati, *Flux quantization and the M-theoretic characters*, *Nucl. Phys.* **B 727** (2005) 461 [hep-th/0507106].

[10] H. Sati, *Duality-symmetry and the form-fields in M-theory*, hep-th/0509046.

[11] M. Ando, M. J. Hopkins, and N. P. Strickland, *Elliptic spectra, the Witten genus and the theorem of the cube*, *Invent. Math.* **146** (2001) 595.

[12] E. Witten, *On flux quantization in M-theory and the effective action*, *J. Geom. Phys.* **22** (1997) 1 [hep-th/9609122].

[13] I. Kriz and H. Xing, *The free loop space and a derivation of E-theory from F-theory*, hep-th/0511011.

[14] A. Strominger, *Open p-branes*, *Phys. Lett.* **B 383** (1996) 44 [hep-th/9512059].

[15] P.K. Townsend, *D-branes from M-branes*, *Phys. Lett.* **B 373** (1996) 68 [hep-th/9512062].

[16] E. Diaconescu, G.W. Moore and D.S. Freed, *The M-theory 3-form and E_8 gauge theory*, hep-th/0312069.

[17] A. Sen, *F-theory and orientifolds*, *Nucl. Phys.* **B 475** (1996) 562 [hep-th/9605150].

[18] A. Sen, *Orientifold limit of F-theory vacua*, *Phys. Rev.* **D 55** (1997) 7345 [hep-th/9702165].

[19] A. Sen, *Orientifold limit of F-theory vacua*, *Nucl. Phys.* **68** *(Proc. Suppl.)* (1998) 92 [hep-th/9709159].

[20] E. Witten, *Solutions of four-dimensional field theories via M-theory*, *Nucl. Phys.* **B 500** (1997) 3 [hep-th/9703166].

[21] N. Seiberg and E. Witten, *Electric - magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [hep-th/9407087].

[22] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099].

[23] T. Banks, M.R. Douglas and N. Seiberg, *Probing F-theory with branes*, *Phys. Lett.* **B 387** (1996) 278 [hep-th/9605199].

[24] D.S. Freed, *K-theory in quantum field theory*, math-ph/0206031.

[25] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles and three-dimensional gauge dynamics*, *Nucl. Phys.* **B 492** (1997) 152 [hep-th/9611230].

[26] J. Evslin and H. Sati, *SUSY vs. E_8 gauge theory in 11 dimensions*, *JHEP* **05** (2003) 048 [hep-th/0210090].

[27] R. Dijkgraaf, E.P. Verlinde and H.L. Verlinde, *BPS quantization of the five-brane*, *Nucl. Phys.* **B 486** (1997) 89 [hep-th/9604055].

[28] J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724 [hep-th/9510017].

[29] E. Witten, *Bound states of strings and p-branes*, *Nucl. Phys.* **B 460** (1996) 335 [hep-th/9510135].

[30] D. Husemöller, *Elliptic curves*, Springer-Verlag, New York, 2004.

[31] J. Silverman, *Advanced topics in the arithmetic of elliptic curves*, GTM 151, Springer, 1994.

[32] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, *Simple singularities and $N = 2$ supersymmetric Yang-Mills theory*, *Phys. Lett.* **B 344** (1995) 169 [hep-th/9411048].

[33] P.C. Argyres and A.E. Faraggi, *The vacuum structure and spectrum of $N = 2$ supersymmetric* SU(N) *gauge theory*, *Phys. Rev. Lett.* **74** (1995) 3931 [hep-th/9411057].

[34] U.H. Danielsson and B. Sundborg, *The moduli space and monodromies of $N = 2$ supersymmetric SO(2r+1) Yang-Mills theory*, *Phys. Lett.* **B 358** (1995) 273 [hep-th/9504102].

[35] A. Brandhuber and K. Landsteiner, *On the monodromies of $N = 2$ supersymmetric Yang-Mills theory with gauge group SO(2n)*, *Phys. Lett.* **B 358** (1995) 73 [hep-th/9507008].

[36] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N.P. Warner, *Self-dual strings and $N = 2$ supersymmetric field theory*, *Nucl. Phys.* **B 477** (1996) 746 [hep-th/9604034].

[37] M.J. Duff and J.X. Lu, *Black and super p-branes in diverse dimensions*, *Nucl. Phys.* **B 416** (1994) 301 [hep-th/9306052].

[38] A. Fayyazuddin and M. Spalinski, *The Seiberg-Witten differential from M-theory*, *Nucl. Phys.* **B 508** (1997) 219 [hep-th/9706087].

[39] C. Vafa, *Evidence for F-theory*, *Nucl. Phys.* **B 469** (1996) 403 [hep-th/9602022].

[40] D.R. Morrison, *TASI lectures on compatification and duality*, hep-th/0411120.

[41] J. de Boer et al., *Triples, fluxes and strings*, *Adv. Theor. Math. Phys.* **4** (2002) 995 [hep-th/0103170].

[42] P.S. Aspinwall, *Some relationships between dualities in string theory*, *Nucl. Phys.* **46** *(Proc. Suppl.)* (1996) 30 [hep-th/9508154].

[43] E.G. Gimon and J. Polchinski, *Consistency conditions for orientifolds and d-manifolds*, *Phys. Rev.* **D 54** (1996) 1667 [hep-th/9601038].

[44] J. Polchinski, S. Chaudhuri and C.V. Johnson, *Notes on D-branes*, hep-th/9602052.

[45] E. Witten, *Phase transitions in M-theory and F-theory*, *Nucl. Phys.* **B 471** (1996) 195 [hep-th/9603150].

[46] E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys.* **B 443** (1995) 85 [hep-th/9503124].

[47] A. Dabholkar, *Ten-dimensional heterotic string as a soliton*, *Phys. Lett.* **B 357** (1995) 307 [hep-th/9506160].

[48] C.M. Hull, *String-string duality in ten-dimensions*, *Phys. Lett.* **B 357** (1995) 545 [hep-th/9506194].

[49] J. Polchinski and E. Witten, *Evidence for heterotic — type-I string duality*, *Nucl. Phys.* **B 460** (1996) 525 [hep-th/9510169].

[50] K. Dasgupta and S. Mukhi, *F-theory at constant coupling*, *Phys. Lett.* **B 385** (1996) 125 [hep-th/9606044].

[51] M.A. Walton, *The heterotic string on the simplest Calabi-Yau manifold and its orbifold limits*, *Phys. Rev.* **D 37** (1988) 377.

[52] W. Nahm and K. Wendland, *A hiker's guide to K3: aspects of $N = (4,4)$ superconformal field theory with central charge $c = 6$*, *Commun. Math. Phys.* **216** (2001) 85 [hep-th/9912067].